Analyzing the Trends of the Modal Age at Death Using the LD Model

Futoshi Ishii*

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Introduction

Recently, the modal age at death has been paid more attention as an indicator of longevity (Horiuchi et al. 2013). Although many studies have discussed the modal age at death, there have been few articles that examined decomposition analyses of the change of the modal age in terms of the shifting and/or the compression of the mortality curve.

The author has proposed the Linear Difference (LD) model that is a shift-type adult mortality model and shown that the model has some advantages for the modeling of adult mortality for Japan and several EU countries compared to the decline-type model such as the Lee-Carter model (Ishii and Lanzieri 2013).

In this paper, we propose a new decomposition method for the modal age at death using the LD model, and give decomposition analyses with the method.

1 Data and Method

1.1 Data

In this paper, we use mortality data for female from 1970 to 2010 by the Human Mortality Database(HMD) and those by the Japanese Mortality Database (JMD) for Japan.

We mainly worked on $m_{x,t}$ functions where x is age and t is a calendar year. We extrapolated the mortality rates above age 110 fitting the two parameter logistic model

$$m_{x,t} = \frac{\alpha_t \exp(\beta_t x)}{1 + \alpha_t \exp(\beta_t x)}$$

that are the same method as used in the HMD (and JMD).

* National Institute of Population and Social Security Research, Tokyo (ishii-futoshi@ipss.go.jp)

1.2 Definition of the LD (Linear Difference) Model

Next, we will state the mathematical formulations putting special emphasis on the log mortality and its inverse functions, and the differential of them by time, as used in Ishii and Lanzieri (2013).

Let $X = [0, +\infty)$ be the space of age and $T = (-\infty, +\infty)$ be the space of time. In the following discussion for modeling mortality, we will work on $\mu_{x,t}$, the hazard function for exact age $x \in X$ at time $t \in T$.

The log hazard function of mortality is expressed by $y = \lambda_{x,t} = \log \mu_{x,t}$, where $y \in Y = (-\infty, +\infty)$ is the value of the function. Then, the set $S = \{(x, t, y) | y = \lambda_{x,t}\}$ determines a surface in $\mathbb{R}^3$, called the *log mortality surface*. This is a conventional representation of the log mortality surface. In this representation, $y = \lambda_{x,t}$ would be considered as the height from the X-T plane in $\mathbb{R}^3$.

Here, we consider another representation of the log mortality surface under a set of assumptions.

We assume that $\lambda_{x,t}$ is a smooth continuous function with respect to x and t defined on $X_0 \times T_0 = [0, \omega] \times [t_0, t_1] \subset X \times T$, where $\omega < +\infty$ is a finite maximum age for mortality models.

For the purpose of modeling *adult* mortality, we can further assume that $\lambda_{x,t}$ exhibits a strictly monotonic increase with respect to x for each t and $x > x_0(t)$. Here, $x_0(t)$ represents the lower bound of x above which $\lambda_{x,t}$ exhibits a strictly monotonic increase for each t. Then, for each t, the function $\lambda_t(x)$ defined by

$$\lambda_t : \tilde{X}_t \longrightarrow Y, \quad \lambda_t(x) \underset{\text{def}}{=} \lambda_{x,t}$$

is an injective (one to one) function of x, where $\tilde{X}_t = [x_0(t), \omega]$. Let $\tilde{Y}_t = \lambda_t(\tilde{X}_t)$, then $\lambda_t(x) : \tilde{X}_t \longrightarrow \tilde{Y}_t$ has an inverse function $\nu_t(y) : \tilde{Y}_t \longrightarrow \tilde{X}_t$ defined on $\tilde{Y}_t$ for each t.

Let us define Y_0 as follows:

$$Y_0 \underset{\text{def}}{=} [y_0, y_1] \quad \text{where} \quad y_0 = \sup_{t \in T_0} \min \tilde{Y}_t, \;\; y_1 = \inf_{t \in T_0} \max \tilde{Y}_t,$$

Then, we can define $\nu_{y,t} : Y_0 \times T_0 \longrightarrow X_0$ by

$$\nu_{y,t} \underset{\text{def}}{=} \nu_t(y)$$

$\nu_{y,t}$ gives the *age* x at which the value of the log hazard function is equivalent to a value y at time t.

Moreover, we define the following two differential functions by time t: (1) $\rho_{x,t}$: the mortality improvement rate and (2) $\tau_{y,t}$: the force of age increase.

$$\rho_{x,t} \underset{\text{def}}{=} -\frac{\partial \lambda_{x,t}}{\partial t} = -\frac{\partial \log \mu_{x,t}}{\partial t}$$

$$\tau_{y,t} \underset{\text{def}}{=} \frac{\partial \nu_{y,t}}{\partial t}$$

Figure 1 shows a stylized example of the log mortality surface and the above two functions. The blue lines show the log mortality surface in a usual representation, that is, the height from the X-T plane which is determined by $\lambda_{x,t}$. The black point on the log mortality surface is $(x,t,y) = (1,2,-1.5)$, which could be recognized that the height from the X-T plane is -1.5 when $(x,t) = (1,2)$. If we travel on the surface with x fixed, the height from the X-T plane will decrease to around -1.86 when $t = 3$, which is shown in a brown arrow. The difference between the two heights corresponds to $-\rho_{x,t}$.

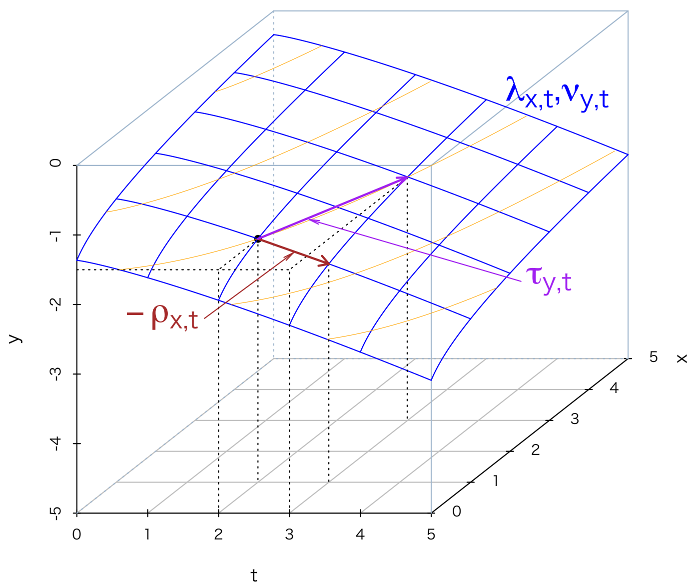
On the other hand, the log mortality surface is also represented by the height from the Y-T plane, which is determined by $\nu_{y,t}$. In this viewpoint, the black point is recognized that the height from the Y-T plane is 1 when $(y,t) = (-1.5,2)$. The orange lines on the surface show the contour with y fixed, so we go along these lines when we travel on the surface with y fixed. If we start from the black point again but keep y fixed this time, the hight from Y-T plane will be 3 when $t = 3$, which is shown in a purple arrow in the figure. The difference between the two heights corresponds to $\tau_{x,t}$.

Let us define the LD model satisfying the property that $\tau_{y,t}$ is a linear function of x for each t, i.e. $\tau_{y,t} = f_t' + g_t' x$. By integrating both sides with t, we obtain $\nu_{y,t} = f_t + g_t x + a_y$ where a_y denotes a standard pattern of inverse log hazard rates.

The Figure 2 shows the stylized example of the LD model. The colored horizontal arrows in the upper half of the Figure 2 show the amount of shifts of the mortality curve indicated with the black line, which correspond to the $\tau_{y,t}$. The vertical arrows at the bottom have same lengths as in the upper side with the same color whose directions are rotated 90 degrees counter-clockwise. The LD model requires that the amount of shifts is a linear function of age, which means the end point of the arrows form a line. The parameter g_t' means the slope of the above line, so the g_t means the slope of age increases between time t and t_0 (base point of time).

Here, we consider another variable S_t as a location of the mortality curve instead of f_t' or f_t. S_t is defined as the age that the mortality rate equals to 0.5 at time t. We can always covert from S_t to f_t using the value of g_t. The Figure 3 shows the stylized example of the effect of change in S_t and g_t. Assuming that the mortality curve at a base point of year is shown as the black line, the increase of S_t with g_t fixed changes the curve into one shown as the red line. Therefore, we can recognize the mortality improvement by the increase of the S_t as the

Figure 1 Log Mortality Surface and Two Differential Functions



shifting of the mortality curve. On the other hand, the decline of g_t with S_t fixed changes the curve into one shown as the blue line, which exhibits some compression features of mortality during the improvement.

1.3 Decomposition of the Change in Modal Age using the LD Model

First, we describe the methods for estimating the M_t, modal age at death. It is often difficult to estimate the M_t from the raw d_x functions in the life tables due to the fluctuations. Therefore, smoothing methods and/or parametric modelings are usually used in estimation. Canudas-Romo (2008) used the approximation by quadratic function originally proposed by Kannisto. Horiuchi et al. (2013) used a nonparametric smoothing method based on P-splines. Here, we used the minimum-R_3 moving averages with 9 terms by Greville (1981) for smoothing the m_x functions that is used in the official life tables for Japan, and estimated the M_t using quadratic approximations used in Canudas-Romo (2008).

In the LD model, we can derive the following decomposition of the trends of M_t: the modal

Figure 2 Stylized example of the LD model

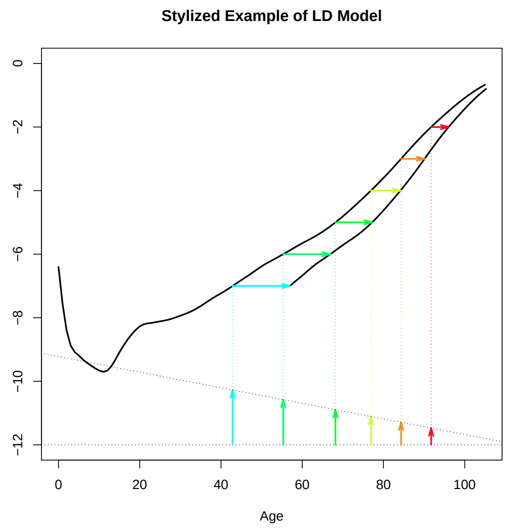
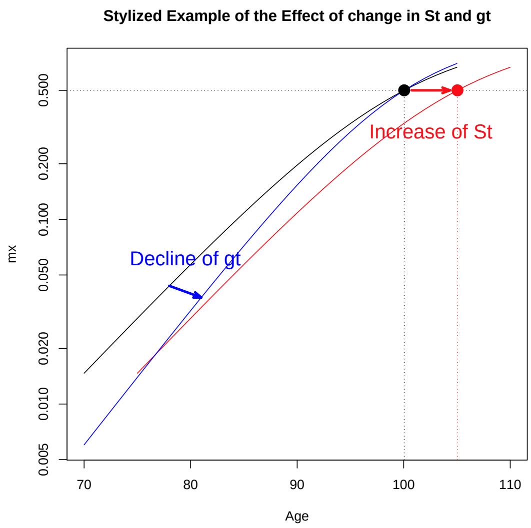


Figure 3 Stylized example of the LD model



age at death.

$$\frac{d}{dt}M_t = f_t' + g_t'\left(M_t - \frac{1}{\mu_{x,t} - \frac{\frac{\partial^2}{\partial x^2}\lambda_{x,t}}{\frac{\partial}{\partial x}\lambda_{x,t}}} \right) = S_t' + g_t'(M_t - S_t) - g_t' D_t$$

where $D_t = \frac{1}{\mu_{x,t} - \frac{\frac{\partial^2}{\partial x^2}\lambda_{x,t}}{\frac{\partial}{\partial x}\lambda_{x,t}}}$.

This formula is interpreted as follows. The S_t' stands for the amount of shifting, the $g_t'(M_t - S_t)$ does for the effect of compression at the modal age, and the $-g_t' D_t$ does for the gap of the modal age at $t + dt$ and the age at $t + dt$ that the value of the $\lambda_{x,t}$ of the modal age at t is taken. Moreover, the formula $\frac{d}{dt}M_t = f_t' + g_t'(M_t - D_t)$ could also be seen as the change of the modal age at death is equal to the force of age increase for the age $M_t - D_t$.

To derive this decomposition, we first notice a relationship that holds on the M_t.

Proposition 1. *When $x = M_t$, then*

$$\frac{\partial}{\partial t}\mu_{x,t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\lambda_{x,t}$$

Proof.

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} l_{x,t} &= -\frac{\partial}{\partial x}(\mu_{x,t} l_{x,t}) \\
&= -\frac{\partial \mu_{x,t}}{\partial x} l_{x,t} - \mu_{x,t} \frac{\partial l_{x,t}}{\partial x} \\
&= -l_{x,t} \left\{ \frac{\partial \mu_{x,t}}{\partial x} - \mu_{x,t}^2 \right\}
\end{aligned}
$$

If $x = M_t$, then $\frac{\partial^2}{\partial x^2} l_{x,t} = 0$. Therefore,

$$
\begin{aligned}
& \frac{\partial \mu_{x,t}}{\partial x} = \mu_{x,t}^2 \\
\Leftrightarrow & \frac{\partial}{\partial x} \log \mu_{x,t} = \mu_{x,t} \\
\Rightarrow & \frac{\partial}{\partial t} \mu_{x,t} = \frac{\partial}{\partial t} \frac{\partial}{\partial x} \lambda_{x,t}
\end{aligned}
$$

□

In the following discussion, we will consider the expression of the M_t as a linear combination of the tangent vectors on S which directions are defined either x or y is fixed. Then we use the above relationship to describe the location of M_t.

Before we write down the formula for M_t, we show the following relationships that hold in the LD model.

Proposition 2. *When x is fixed,*

$$
\frac{\partial}{\partial t} \mu_{x,t} = -\frac{\partial}{\partial x} \mu_{x,t} (f'_t + g'_t x)
$$

$$
\frac{\partial}{\partial t} \left(\frac{\partial}{\partial x} \lambda_{x,t} \right) = -\frac{\partial^2}{\partial x^2} \lambda_{x,t} (f'_t + g'_t x) - \frac{\partial}{\partial x} \lambda_{x,t} g'_t
$$

Proof. The log mortality surface S is defined by the equation $\lambda_{x,t} - y = 0$, the tangent space on (x_0, t_0, y_0) is $Y - y_0 = \frac{\partial \lambda_{x,t}}{\partial x}(X - x_0) + \frac{\partial \lambda_{x,t}}{\partial t}(T - t_0)$. Now $(\tau_{y,t}, 1, 0)$ is a tangent vector on S, therefore we have

$$
\begin{aligned}
& 0 = \frac{\partial \lambda_{x,t}}{\partial x} \tau_{y,t} + \frac{\partial \lambda_{x,t}}{\partial t} \\
\Leftrightarrow & \frac{1}{\mu_{x,t}} \frac{\partial \mu_{x,t}}{\partial t} = -\frac{1}{\mu_{x,t}} \frac{\partial \mu_{x,t}}{\partial x} (f'_t + g'_t x)
\end{aligned}
$$

This shows the first equation.

Then, using this formula,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\lambda_{x,t}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\lambda_{x,t}\right) \\
&= -\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\lambda_{x,t}(f_t' + g_t' x)\right\} \\
&= -\frac{\partial^2}{\partial x^2}\lambda_{x,t}(f_t' + g_t' x) - \frac{\partial}{\partial x}\lambda_{x,t} g_t'
\end{aligned}$$

This completes the proof of the second equation. □

Next we observe the similar relationship when y is fixed. Obviously, $\frac{\partial}{\partial t}\mu_{x,t} = 0$ when $y = \lambda_{x,t}$ is fixed. We consider the directional derivative along $\tau_{y,t}$ of the slope $\frac{\partial}{\partial x}\lambda_{x,t}$.

Proposition 3. *When y is fixed,*

$$\frac{\partial}{\partial t}\mu_{x,t} = 0$$

$$D_{\tau_{y,t}}\left(\frac{\partial}{\partial x}\lambda_{x,t}\right) = -\frac{\partial}{\partial x}\lambda_{x,t} g_t'$$

Proof. The slope at $t = t_0$ is expressed as $\frac{\triangle y}{\triangle x}$ when $\triangle x \to 0$. Then, at $t = t_0 + h$ for small h, the slope is expressed as

$$\frac{\triangle y'}{\triangle x'} = \frac{\triangle y}{(1 + g_{t_0}')h\triangle x}$$

since $\triangle x' = \{x_0 + \triangle x + (f_{t_0}' + g_{t_0}'(x_0 + \triangle x))h\} - \{x_0 + (f_{t_0}' + g_{t_0}'(x_0))h\}$.

Then,

$$\begin{aligned}
D_{\tau_{y,t}}\left(\frac{\partial}{\partial x}\lambda_{x,t}\right) &= \lim_{h\to 0}\frac{1}{h}\left\{\frac{\triangle y}{(1 + g_{t_0}')h\triangle x} - \frac{\triangle y}{\triangle x}\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\frac{\triangle y}{\triangle x}\frac{-hg_{t_0}'}{1 + hg_{t_0}'} \\
&= -\frac{\partial}{\partial x}\lambda(x_0, t_0) g_{t_0}'
\end{aligned}$$

□

Now we are ready to derive the first formula. Assume that **M**: the tangent vector on S along $x = M_t$ is expressed by a linear combination of **A** and **B** as $\mathbf{M} = (1 - k)\mathbf{A} + k\mathbf{B}$, where **A** and **B** are the tangent vectors on S when x and y is fixed respectively. Then $\frac{d}{dt}M_t = k(f_t' + g_t' x)$.

Using the Proposition 1,

$$-(1-k)\left\{\frac{\partial^2}{\partial x^2}\lambda_{x,t}(f_t'+g_t'x)-\frac{\partial}{\partial x}\lambda_{x,t}g_t'\right\}-k\frac{\partial}{\partial x}\lambda_{x,t}g_t'=-(1-k)\frac{\partial}{\partial x}\mu_{x,t}(f_t'+g_t'x)$$

$$\Leftrightarrow k(f_t'+g_t'x)=-\frac{g_t'}{\mu_{x,t}-\frac{\frac{\partial^2}{\partial x^2}\lambda_{x,t}}{\frac{\partial}{\partial x}\lambda_{x,t}}}+(f_t'+g_t'x)$$

Substituting $x=M_t$,

$$\frac{d}{dt}M_t=f_t'+g_t'\left(M_t-\frac{1}{\mu_{x,t}-\frac{\frac{\partial^2}{\partial x^2}\lambda_{x,t}}{\frac{\partial}{\partial x}\lambda_{x,t}}}\right)=S_t'+g_t'(M_t-S_t)-g_t'D_t$$

where $D_t=\frac{1}{\mu_{x,t}-\frac{\frac{\partial^2}{\partial x^2}\lambda_{x,t}}{\frac{\partial}{\partial x}\lambda_{x,t}}}$.

This completes the proof of the decomposition.

2 Results

Here, we show the results for Japan. Figure 4 shows the actual log mortality rates and Figure 5 show the trends of S_t and g_t. We can observe that the S_t has increased steadily in this period, that reveals the shifting feature of the old mortality. On the other hand, g_t has remained almost stationary from 1980 to 2000 that means the shifting is strongly close to parallel shift in this period, although it has decreased before 1980 and after 2000.

Figure 4 Mortality Rates (Actual, Female, Japan)

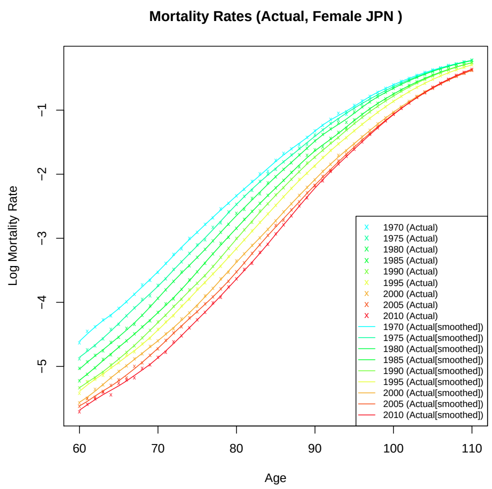


Figure 5 Trends of S_t and g_t(Female Japan)

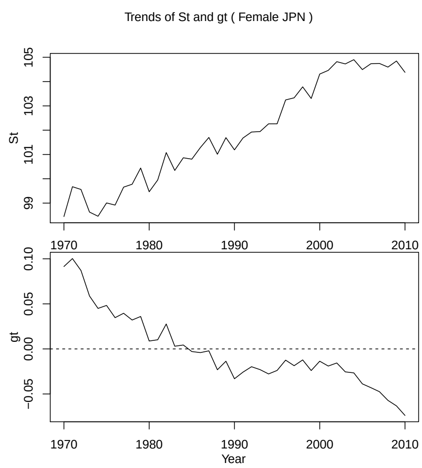


Figure 6 shows the trends of the M_t for the actual mortality and the LD model. We can observe that the both trends are similar, so we analyze with the mortality rates by the LD model.

Figure 7 shows the results of the decomposition results of the change of M_t for the LD model by every ten years. We can observe that the increase of M_t is mainly caused by the shifting from 1980 to 2000, whereas compression plays a larger part before 1970 and after 2000.

Figure 6 Trends of the Modal Age at Death (Actual and LD, Female, Japan)

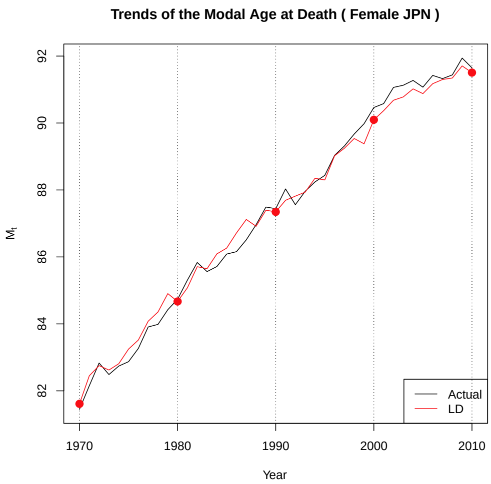
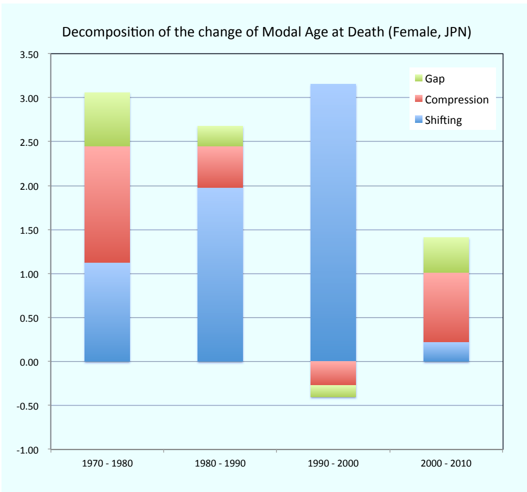


Figure 7 Decomposition of the change of Modal Age at Death (Female Japan)



3 Concluding Remarks

In this paper, we proposed a new decomposition method for the modal age at death using the LD model. The LD model is originally developed for mortality projections for Japan. Therefore, the number of parameters are reduced in terms of parsimony. This might be a restriction in terms of a flexible expression for various types of mortality situation. Actually, Ishii and Lanzieri (2013) has shown that the LD model works very well for some EU countries, whereas it does not much for others.

However, this feature brings another possibility to derive simple analytical formula, as we have just discussed in this paper. The decomposition that we propose is easy to apply when the mortality curves are modeled by the LD model, and has a clear interpretation composed by shifting, compression and other parts. We have shown the decomposition analysis for Japanese female as an example. From the results, we observed a strong parallel shifting feature from 1980 to 2000 that also increased M_t by shifting components. On the other hand, the compression components played a larger part for the increase of M_t before 1970 and after 2000.

The analytical decomposition of trends in the modal age at death proposed in this paper would be considered useful for understanding of old age mortality. At the same time, we have also made it clear that the LD model has various applicability other than mortality projections.

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